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Vol. XI BATON ROUGE, LA., MARCH, 1937 No. 6

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1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

"Let Us Not Surrender"

February 14th, 1937.

My dear Professor Sanders:

I have naturally read with interest your two editorials in the National Mathematics Magazine, dated December, 1936 and January, 1937. .... All the objections to the actual state of affairs that you advance are justified. We all know the unsatisfactory general situation in High School Mathematics. We also know, as you point out, that in the Classics, and also in modern Languages, exactly what you are urging for Mathematics has actually taken place.

Yet it seems to many among us who are vitally interested in these questions, that the path you are proposing leads in the wrong direction.

Speaking only for myself, I should object for the following reasons:

I. The present state of affairs, and the tendency of today to take all serious academic work out of the High School and put it into the University and College, does not represent a possible permanent state of affairs. The utter absurdity of allowing our boys and girls to reach the age of around eighteen without having any foundation for a real education is so obvious that, sooner or later, the pendulum is bound to swing in the opposite direction, provided enough among us—both in the secondary School System and in the Colleges and Universities—are determined to stand for what we know is necessary.

II. America has undertaken a larger task than it can at present accomplish. The time required to build up a secondary school system capable of dealing adequately with the millions of high school students must be measured in terms of generations. But, if we push elementary Mathematics over into the Colleges, it will mean that the great majority of high school students will have no Mathematics beyond Arithmetic; and even so, the additional burden thrown on to the institutions of higher learning will lower our own standards, both on account of the probable vast increase in teaching load, and because of the low level of the work we shall be doing.

III. .... Rather, it would seem to be a sound policy to organize our forces in both the elementary and advanced educational systems against the destructive influences, to put forward the best efforts of all men and women concerned in our problems to formulate, and to promote by all means at our disposal, policies which will strengthen our fellow workers in the secondary system.

Yours very sincerely,

AUBREY J. KEMPNER,
University of Colorado.

The Octo-Binary System

By G. W. WISHARD
Norwood, Ohio

The decimal notation will not be replaced by any other in our time, and furthermore it may never be superseded. But notwithstanding all this, a research of the various number systems would be both interesting and instructive—much more so than is generally supposed. They contain many curiosities that would throw light on mathematics. Their investigation has not been so thorough as many think.

For instance, the advocates of the octonary system overlook a very important property. In the "*Proceedings of the American Pharmaceutical Association*" for 1859, Alfred B. Taylor, of Philadelphia, set forth a long and interesting "*Report of the Committee on Weights and Measures*". It is apparent that he was the chief member of the committee; and that he had read everything he could find on the subject; and furthermore that he had done a great deal of thinking.

But after paying a grand tribute to the binary system for its simplicity, beauty and facility, he threw it entirely overboard, because it is too voluminous for general use.

He chose the scale of 8, because it is a perfect cube, and also suited for doubling and halving, so often used in metrology. Then he developed an octonary arithmetic, weights and measures.

He did not perceive how easily the octonary can be changed into the binary, and the binary into the octonary; or he would certainly have salvaged a large part of the binary benefits.

Since $8=2^3$, each octonary figure is equal to three binary figures, thus:

Octonary		*Binary*	*Octonary*		*Binary*
0	=	000	4	=	100
1	=	001	5	=	101
2	=	010	6	=	110
3	=	011	7	=	111

This is capable of algebraic proof:

$$r^3)\overline{ar^8+br^7+cr^6+dr^5+er^4+fr^3+gr^2+hr+i}$$
$$r^3)\overline{ar^5+br^4+cr^3+dr^2+er+f} \quad Rem=gr^2+hr+i$$
$$\overline{ar^2+br+c} \quad Rem=dr^2+er+f$$

Consequently, each octonary order is equal to three binary orders.

By remembering the above table of 8 items, an octonarist could change any octonary number to the binary as fast as he could write the figures, and that without any computation whatever. Then he could double or halve it repeatedly by simply moving the binary point, and filling the vacant places with ciphers.

He could perform many calculations more easily in the binary than in the octonary. Although the binary would contain three times as many figures, yet half of them would, on the general average, be 0's, quickly made and passed over in calculation. He could often employ some short method. He could use slide multiplication to the best advantage, and it would be unnecessary to write down the partial products.

He could easily and quickly change the result back to the octonary for record and general nomenclature.

If by any miracle, the Octo-Binary ever came into general use, the accountants would, of course, learn to compute in both scales, and be able to use the most expeditious one in the particular solution.

Since the octo-binary can be so easily folded and unfolded, it ought to arouse the curiosity of the students in the theory of numbers and in mathematical clubs. It has a beautiful series of squares, and the odd squares* plainly show the triangular numbers, which have many interesting properties.

It might be amusing to try the lowly binary. Leibnitz regarded it as a splendid instrument of investigation. It goes down to the very roots of the quaternary, octonary and sexadecimal notations, and it contains almost all of their properties except brevity.

Any grand numeric reform is hopeless now, and will continue to be so, until some very strong X-Rays of research be applied to the various number systems to find out what secrets and possibilities are in them. No two systems are the same, and no one contains all the good points. No mere superficial inspection can determine the best scale, and how to use it.

*National Mathematics Magazine, May, 1936, page 313.

Generalization of Pascal's Arithmetical Triangle

By ALTHÉOD TREMBLAY
Laval University, Quebec, Can.

I.

Demoivre's formula* will give the coefficient of x^p in the expansion of:

$$(x^1+x^2+x^3+\ldots\ldots x^n)^m$$

It is remarkable that the computation of the same coefficient can be easily made by forming a table similar to Pascal's Arithmetical Triangle, which gives the coefficients of the expansion of a binomial.

The rule for forming a table of the coefficients of the terms of the mth power of a polynomial of n terms, all the coefficients being unity, is as follows:

Write unity n times in the first row.

1 1 1 1 1 . . . .

Then a term in any row of the table is formed by the addition of n terms of the preceding row, beginning with the term above the one sought and adding to it the $(n-1)$ preceding terms.

The first row of such a table shows the coefficients of the terms of the 1st power of the given polynomial, all equal to unity. The second row gives the coefficients of the terms of the 2nd power, the third row those of the 3rd power etc.... the mth row will be the expansion of the mth power.

When applied to a binomial this method will give Pascal's Arithmetical Triangle.

1 1
1 2 1
1 3 3 1
. . . . .

The coefficients of the expansion of the trinomial $(x^1+x^2+x^3)^m$ are:

1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
. . . . . .

*See Hall, H. S. and Knight, S. R., *Higher Algebra*, p. 389.

Those of the quadrinomial $(x^1+x^2+x^3+x^4)^m$ are:

1	1	1	1								
1	2	3	4	3	2	1					
1	3	6	10	12	12	10	6	3	1		
.	.	.	.	.	.	.	.	.	.	.	.

For the hexanomial $(x^1+x^2+x^3+x^4+x^5+x^6)^m$ we get

1	1	1	1	1	1								
1	2	3	4	5	6	5	4	3	2	1			
1	3	6	10	15	21	25	27	27	25	21	15	10	6
1	4	10	20	35	56	80	104	125	140	146	140	125	104
.	.	.	.	.	.	.	.	.	.	.	.	.	.

If ${}_nK_m^p$ stands for the coefficient of x^p in $\left(\sum_{i=1}^{n} x^i\right)^m$ the above rule may be formulated thus:

$$ {}_nK_{m+1}^p = \sum_{i=1}^{n} {}_nK_m^{p-i} $$

The coefficients of the last table have some importance in the calculation of probabilities in dice throwing.

The numerator of the probability of turning the point 12 with 4 dice, for instance, is the coefficient of x^{12} in the expansion of

$$(x^1+x^2+x^3+x^4+x^5+x^6)^4.$$

It is the 9th term of the 4th line of the last table, 125. The probability is

$$P = 125/6^4$$

The formation of these tables is very easily proved by mathematical induction. Simple as it may be it is surprising to find out how this method helps solving questions of probabilities in dice throwing. One can detect at a glance many obvious relations which the usual formulae could reveal only with much labor.

For instance the equality of the coefficients equally distant from both ends and the fact that the middle or two middle terms are maximum, will solve rapidly the following problem.

How many dice should be taken so that the turning of the point 35 in one throw should have the greatest probability?

The number of dice required is between 6 and 35 included. To solve this problem we have to find m in the expansion of:

$$(x^1+x^2+x^3+x^4+x^5+x^6)^m$$

so that the mth power will have x^{35} for its middle term. The sum of the exponents of x in terms equally distant from both ends is constant and equal to $6m+m$. Then:

$$6m+m=70$$

$$m=10$$

The number of dice required is 10. (If the quotient is not an integer, it can be proved that, in general, the nearest integer greater or smaller than the quotient will give the required result.)

II.

Pascal's Arithmetical Triangle may be extended upwards, that is above the first row, by the same method that one would use to compute the terms of a row with the help of the terms of the following row, and the terms preceding the one sought.

The rows thus formed will give the coefficients of the infinite series, being the expansions of the negative powers of the binomial.

$(x+1)^{-3}$	:	1	−3	6	−10	15
$(1+x)^{-2}$	:	1	−2	3	−4	5
$(1+x)^{-1}$	:	1	−1	1	−1	1
$(1+x)^{0}$	:	1	0	0	0	0
$(1+x)^{1}$	:	1	1			
$(1+x)^{2}$	:	1	2	1		

We may notice that the coefficients of the expansion of $(1+x)^{-m}$ are the figurate numbers of the mth order alternately postive and negative.

The same remark applies to the expansion of the negative powers of

$$(1+x+x^2+\ldots\ldots x^n)^{-m}$$

The tables in part I which may be called *Arithmetical Trapeziums*, giving the coefficients of the positive powers of polynomials, may be extended upwards to give the negative powers of the same polynomials, by a rule which is easily deduced from the law of formation of a term using the $(n-1)$ preceding terms and a term in the following row.

With the trinomial we get:

$(1+x+x^2)^{-2}$	:	1	−2	1	2	−4
$(1+x+x^2)^{-1}$	:	1	−1	0	1	−1
$(1+x+x^2)^{0}$	:	1	0	0	0	0
$(1+x+x^2)^{1}$	:	1	1	1	0	0
$(1+x+x^2)^{2}$	:	1	2	3	2	1

On the Difference Between Two Sample Variances*

By ALLEN T. CRAIG
The University of Iowa

When two independent sets of observations on a variable are available, it is an important statistical problem to determine, under appropriate restrictions, whether a function T of the observations of the first set differs significantly from that function T of the observations of the second set. In this paper, we shall be concerned with certain aspects of this problem when T is the variance of the observations.

Let the variable x be subject to the Gaussian Law of Error

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

and let $x_1, x_2, \cdots, x_N$ and $x_1', x_2', \cdots, x'_N$ be two independent samples of N items each. Write $Ns_1^2 = \sum(x_j - \bar{x}_1)^2$ and $Ns_2^2 = \sum(x'_j - \bar{x}_2)^2$ where $N\bar{x}_1 = \sum x_j$ and $N\bar{x}_2 = \sum x_j'$. The variables

$$u = \frac{Ns_1^2}{\sigma^2} \quad \text{and} \quad v = \frac{Ns_2^2}{\sigma^2}$$

are independent in the probability sense and are distributed in accordance with

$$g(u) = Cu^{\frac{N-3}{2}} e^{-\frac{u}{2}}$$

and

$$h(v) = Cv^{\frac{N-3}{2}} e^{-\frac{v}{2}}$$

respectively, where

$$1/C = 2^{\frac{N-1}{2}} \Gamma\left(\frac{n-1}{2}\right).$$

We write $y = u - v$ and seek the distribution function $F(y)$ of the difference y.

*Presented to the Iowa Section of the Mathematical Association of America, April 4, 1936.

The characteristic function $\varphi_1(t)$ of the distribution of u is given by

$$\varphi_1(t) = \int_0^\infty e^{itu} g(u)du = (1-2it)^{-\frac{N-1}{2}}, \quad i = \sqrt{-1},$$

and that of the distribution of $-v$ is given by

$$\varphi_2(t) = \int_0^\infty e^{-itv} h(v)dv = (1+2it)^{-\frac{N-1}{2}}.$$

Accordingly, the characteristic function of the distribution of y is

$$\varphi(t) = \varphi_1(t)\varphi_2(t) = (1+4t^2)^{-\frac{N-1}{2}}.$$

By the Fourier Integral Theorem, we see that

$$F(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ity}\, \varphi(t)dt.$$

In case N is an odd integer, $N = 2k+1$, we may evaluate the above integral by an appropriate choice of contour for integration in the complex plane. We find that

$$F(y) = \frac{1}{2} e^{-\frac{1}{2}|y|} \left[a_0 + \frac{a_1}{2}|y| + \frac{a_2}{2^2} y^2 + \cdots + \frac{a_{k-1}}{2^{k-1}} |y^{k-1}| \right]$$

where

$$a_r = \frac{\underline{|2k-r-2}}{2^{2k-r-1}\underline{|k-r-1}\,\underline{|r}\,\underline{|k-1}}, \qquad r = 0,1,\cdots,k-1.$$

Thus, the probability P that

$$s_1^2 - s_2^2 = \frac{\sigma^2 y}{2k+1}$$

lies within an interval $(-\lambda\sigma^2, \lambda\sigma^2)$ is given by

$$P = 2\int_0^{\frac{\lambda(2k+1)}{2}} e^{-z}[a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}]dz.$$

It is interesting to observe that P depends only on the number of items in the samples and the assigned multiple λ of the population

variance. It is evident that Pearson's *"Tables of the Incomplete Gamma Function"* may be used to approximate the probability.

If $N=2k+1$ is a large number, we may appropriately make use of the asymptotic values of the coefficients a_r. It is easily seen that

$$a_0=a_1\sim\frac{1}{2\sqrt{\pi(k-1)}},\quad a_r\sim\frac{1}{\lfloor r}a_0,\quad r=2,3,\cdots,k-1$$

Accordingly, for large values of N, we may write

$$P=\frac{1}{\sqrt{\pi(k-1)}}\int_0^{\frac{\lambda(2k+1)}{2}} e^{-z}\left[1+z+\cdots+\frac{z^{k-1}}{\lfloor k-1}\right]dz$$

approximately.

For the case in which N is an even integer, we may usefully represent $F(y)$ by a Gram-Charlier Type A series. If w_s is the sth semi-invariant of Thiele of the distribution of y, it follows from

$$i^s w_s=\left.\frac{d^s \log \varphi(t)}{dt^s}\right|_{t=0}$$

that $w_{2s+1}=0$ and $w_{2s}=4^s\lfloor 2s-1$ $(N-1)$. In terms of the semi-invariants, the distribution function $F(y)$ is given by

$$e^{\sum_1^\infty \frac{(it)^s}{\lfloor s}w_s}=\int_{-\infty}^{\infty} e^{ity}F(y)dy.$$

Thus,

$$F(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ity+\sum_1^\infty \frac{(it)^s}{\lfloor s}w_s}dt$$

$$=\psi(y)+A_2\psi^{(2)}(y)+A_4\psi^{(4)}(y)+\cdots$$

where

$$\psi(y)=\frac{1}{\sqrt{2\pi w_2}}e^{-\frac{y^2}{2w_2}},$$

$$\psi^{(m)}(y)=\frac{d^m\psi(y)}{dy^m},$$

and the A's are the well known functions of the semi-invariants.† Accordingly, the probability that $s_1^2-s_2^2$ lies within an interval $(-\lambda\sigma^2,\lambda\sigma^2)$ is given, whether N be even or odd, by

$$P=2\int_0^{\lambda N}[\psi(y)+A_2\psi^{(2)}(y)+\cdots]dy.$$

It is clear that P depends only on N and λ. It is further evident that tables of the Gaussian Law of Error and its derivatives are adequate to approximate P.

†Cf. T. N. Thiele, *Theory of Observations* (1903) pp. 33-35.

JOHANN FRIEDRICH PFAFF

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Johann Friedrich Pfaff

By G. WALDO DUNNINGTON
University of Illinois

Pfaff was descended from an old middle class family which goes back to a Swiss who at the beginning of the sixteenth century moved from Aarau to Württemberg for religious reasons. His father was Friedrich Burkhard Pfaff (1738-1817), holding the title of "Geheimer Oberfinanzrat", a public official in Stuttgart noted for his diligence, insight, and skill. His mother was a daughter of the "Kirchenrat" Brand; they had seven sons and five daughters. Three of the sons were simultaneously professors in German universities. The second of these three was Christoph Heinrich Pfaff (1773-1852), well known professor of medicine and chemistry at the University of Kiel. The youngest was also a mathematician, who will be mentioned at the conclusion of this article.

Johann Friedrich was born December 22, 1765, in Stuttgart as the second of Burkhard Pfaff's seven sons. At the age of nine he was sent to the famous ducal Carl's Academy, which was naturally open to the son of an official. Schiller attended this academy in 1773, although he was at the time already fourteen years old. Pfaff was a pupil there from 1774 to 1785; among his teachers were Drück, Moll, and Nast. Duke Carl became greatly interested in Johann Friedrich, who was taking and actually completed, the law course. His real talent was a mathematical one, and this did not escape the attention of his teachers. Among his fellow students were Carl Friedrich Kielmeyer, later a notable scientist, and Ludwig Schubart, who was later a journalist and diplomat of some renown. The most coveted honor at the Carl's Academy was that of *chevalier*. Pfaff and Kielmeyer received the decorative cross of this order from the duke's hands on December 22, 1782, the seventeenth birthday of the former. Pfaff was honored with twenty different prizes during his stay at the academy. He was not one-sided, but gave attention to the classics, philosophy, and to German *belles lettres*.

In the fall of 1785 Pfaff was sent by the duke on a scientific trip. He decided to study in Göttingen, remaining there approximately two years, always heeding the duke's desires. In Göttingen he studied under the two luminaries of the period: Käastner and Lichtenberg; under Spittler, classical philosophy with Heyne, and chemistry with Gmelin. He became a close friend of Bouterwek, Buttmann, and Baron Georg Sartorius von Waltershausen, the historian and friend of Goethe. Pfaff carried twenty-two letters of introduction to Göttingen; there were many former residents of Württemberg living at that time in the town. At Kästner's Jubilee Pfaff wrote an eloquent Latin congratulation and sent it to him.

His *Commentatio de ortibus et occasibus siderum apud auctores classicos commemoratis* received the prize of the Göttingen philosophical faculty in 1786. In the summer of 1787 he went to Berlin in order to study practical astronomy under Bode; he also worked there with Merian. Lichtenberg had given him a letter to Nicolai, and he also became acquainted there with E. G. Fischer. His special work in Berlin was the completion of his treatise: *Versuch* einer neuen Summations = Methode nebst andern damit zusammenhängenden analytischen Bemerkungen*, 1788, (published in Berlin). This memoir assured him permanent recognition; it is dedicated to Duke Carl. In 1793 Pfaff was elected a member of the Russian Academy. While in Berlin he became interested in Kepler and planned to write a biography of him, but this never came to fruition. Pfaff reported regularly to Duke Carl during this period and there exists a considerable body of correspondence passed between the two.

From Berlin the Duke sent him to Vienna; he traveled via Halle, Jena, Helmstedt, Gotha, Dresden, and Prague. In Dresden he met Schiller's friend Körner, and in Vienna he was received by the Württemberg ambassador von Bühler. Kästner desired Pfaff as his successor in Göttingen, and after Kastner's death in 1800 he was called there, but he declined, and proposed Gauss. At Easter 1788 Klügel at the University of Helmstedt was called to Halle, and upon Lichtenberg's recommendation the position was offered to Pfaff. With Duke Carl's approval, he accepted. In Helmstedt Pfaff performed valuable services in administration of the University's annuity fund for professors' widows (as later in Halle). In 1802 Pfaff received a call to the University of Dorpat; he declined and proposed his younger brother,

*In Gauss' copy of this memoir I found in his entry on the fly-leaf his name and the date 1793. He had written down the dates and places of Pfaff's birth and death; following a favorite practice of his, he had calculated the number of days Pfaff lived, and found it to be 21,673. We know that he did this for Newton, Alexander von Humboldt, and for any number of famous men, friends, relatives, and contemporaries. There was also a quotation from Euler here.

who received the position. He was financially rewarded for refusing. The duke of Brunswick made him a Hofrat in 1802.

In Helmstedt his special friends were G. G. Bredow, the historian, and P. J. Bruns, the professor of literature. Bruns and Pfaff were transferred in 1810 to the University of Halle, because of the closing of Helmstedt. Pfaff remained in Halle for the remainder of his life. Among his friends were also Gauss, Gerling, Alexander von Humboldt, and G. E. Schulze, the philosopher, who was transferred to Göttingen in 1810.

Pfaff received his doctorate in 1788 from the Carl's Academy in Stuttgart; in the same year appeared his *Programma inaugurale, in quo peculiaris differentialia investigandi ratio ex theoria functionum deducitur* (Helmstedt). In 1797 he published a paper with the Russian Academy *Observationes analyticae ad L. Eulerii institutiones calculi integralis* in the Acta acad. scient. Petropol. T. XI, Histoire, pp. 35-57. Hindenburg's *Archiv der Mathematik* and *"Sammlung"* 1794-1800 contains nine papers by Pfaff. He published in 1797 an important volume *Disquisitiones analyticae maxime ad calculum integralem et doctrinam serierum pertinentes,* in which summation of series, Lagrange's theorem, the polynomial theorem, and linear differential equations of the second order are treated, a preliminary to his greatest achievement. In 1810 Pfaff contributed in Zach's *Monatliche Correspondenz* to the solution of Gauss' problem on the ellipse of maximum area inscriptible in a given quadrilateral.

His most important memoir appeared in 1815 in the transactions of the Berlin Academy: *Methodus generalis aequationes differentiarum particularum, nec non aequationes differentiales vulgares, utrasque primi ordinis, inter quotcunque variabiles, complete integrandi.* J. T. Mayer incorporated the substance of this method in a textbook appearing in 1818. According to Moritz Cantor, it was made known generally in Europe by Jacobi's paper of 1827 in Crelle's Journal, v. II, p. 347: *Über die Pfaff'sche Methode eine gewöhnliche lineare Differentialgleichung zwischen 2n Variabeln durch ein System von n Gleichungen zu integriren.*

Pfaff contributed to Bredow's *Chronik des 19. ten Jahrhunderts* a work on the calendar. In 1796 he published in another field *Über die Vortheile, welche eine Universität einem Lande gewährt* (Häberlin's *Staatsarchiz*). His honors from learned bodies extended over a period of 28 years:

1793—corresponding member, Academy of St. Petersburg.
1793—corresponding member of the Royal Society of Sciences in Göttingen.

1798—full membership in the Russian Academy, mentioned above.
1801—member of the *deutsche Gesellschaft* in Helmstedt.
1811—member of the *Gesellschaft Naturforschender Freunde in Halle.*
1812—corresponding member of the Berlin Academy of Sciences.
1817—full membership in the Berlin Academy.
1821—corresponding member in the *Institut de France.*

In 1803 Pfaff married Caroline Brand (b. March 28, 1784), a cousin, the eldest daughter of Rev. Christoph Brand in Mussberg near Stuttgart. Their son, Dr. Carl Pfaff, b. February 1, 1805, in Helmstedt, was for some years professor of philosophy and literary history at the University of Halle. A second son, Ludwig Pfaff, was born in Halle on December 12, 1811, and died there February 4, 1829.

Apparently Pfaff refused a second call to Göttingen in 1810, when the University of Helmstedt was closed. In Halle he became full professor of mathematics and director of the observatory at Klügel's death in 1812. Pfaff died there of apoplexy on April 20, 1825. Most of his students were later university professors. Among these may be mentioned: Gerling, Gauss, Kämtz, Mollweide, Dirksen, Focke, Möbius, Gartz, Grunert, Schön, Wex, Junge, A. W. Bobtz, J. G. Mussmann, Yxem, Bartels, and the Russians Carzov and Tschischof. He usually had students living in his large home, and enjoyed social intercourse with them. He would stimulate them to prepare addresses commemorative of such men as Kant, Leibniz, or Copernicus. He caused the university library in Halle to be opened daily, and had it made accessible to members of the university for use in the forenoons. His literary interest manifested itself in a club for the reading of foreign journals. In Halle Pfaff's friends were Seffens, Blanc, Bruns, Schütz, Seidler, and Jacobs.

Kant and Fichte were his favorite philosophers; Hegel and Schelling, although Swabians like himself, could not satisfy him with their philosophical systems. In speaking he never lost his Swabian dialect and in the summer usually made a trip to this native district. Pfaff was, like many another mathematician, deeply interested in poetry and politics. He enjoyed the historical and aesthetic writings of the French, and loved to hear poetry read aloud. Schiller, for whom he had a deep admiration and for whose poetry he had a splendid understanding, once received him in Weimar. He called Ludwig Tieck, famous German Romanticist, the founder of modern poetry.

(NOTE—Among the scholars with whom Pfaff corresponded were Biot, Carnot, and Fuss. A list of his works is to be found in the *Neuer Nekrolog der Deutschen,* Vol. 3, 1825, p. 1418.)

WILHELM PFAFF

Johann Wilhelm Andreas Pfaff, son of Burkhard Pfaff, a Commander of the Württemberg Civil Merit Order, and youngest brother of Johann Friedrich Pfaff, was born December 5/8, 1774, in Stuttgart and died June 25, 1835, in Erlangen. He was characterized by vivacity and restlessness, was educated in the Stuttgart secondary schools and at the age of 16 entered the theological seminary in Tübingen, where he received his degree, and took second place in his class. After traveling for some time he became in 1800 *Repetent* at the theological seminary in Tübingen, and in August, 1803, accepted the title of Imperial Russian Hofrat, becoming professor of mathematics in the new University of Dorpat. Although only thirty years old he was director of the observatory there. In September, 1804, he married a lady of the Livonian nobility, Pauline von Patkul, of the Imperial *Fräuleinstift* in Dorpat.

He was dissatisfied in Dorpat and in August, 1809, accepted a call to the *Realinstitut* in Nuremberg; in February, 1817, he became professor of mathematics at the University of Wurzbürg, and in 1818 at the University of Erlangen. His wife died March 15, 1816, in Nuremberg, two sons having preceded her in death; in October, 1817, he married Luise Plank, widow of a preacher, "Oberhelfer" Kraz in Kirchheim, who died in 1814, by whom he had several children. Alexius Burkhard Immanuel Friedrich Pfaff (1825-1886), the well known geologist and mineralogist at the University of Erlangen, active also as a Christian Socialist, was a son of this marriage. A daughter by the first marriage died single December 14, 1832. He died in 1835 after several apoplectic attacks.

Although he produced practically nothing worthwhile, he was a member of the scientific academies at St. Petersburg, Munich, and the physico-medical Society at Moscow. He dabbled in comparative philology, Sanscrit, Egyptian archaeology and hieroglyphic remains, and astrology—which he planned to restore to the status of a science. Bode's *Jahrbuch*, Zach's *monatliche Correspondenz*, and the *Denkschriften der Münchener Akademie* contain some worthwhile calculations of perturbations by him. A list of his writings may be found in the *Neuer Nekrolog der Deutschen*, vol. 13, 1835, pp. 575-578.

The Teachers' Department

Edited by
JOSEPH SEIDLIN

An Experiment in Cooperative Teaching

By J. H. WEAVER
Ohio State University

For a number of years there has been a feeling of dissatisfaction on the part of members of the faculties of the technical engineering departments because students who had taken courses in college algebra, trigonometry, analytic geometry and calculus, were not able to satisfactorily use the facts learned in these courses when taking certain technical engineering courses. This dissatisfaction has led to the folowing suggestions:

1) That all mathematics required of engineering students should be taught by engineers, who could presumably give the proper engineering flavor to the topics taught.
2) That mathematics instructors who teach engineering students should be directly responsible to the college of engineering in order that the college could require them to teach the mathematics needed by engineers in the way engineers want it taught.

These proposals, in one form or another, have appeared a number of times in the College of Engineering at the Ohio State University. In fact it was such a proposal which led to the present experiment. The procedure was as follows.

On May 9, 1934 the Department of Electrical Engineering submitted to the faculty of the College of Engineering a request that the Department of Electrical Engineering be granted permission to offer to students of electrical engineering, on an experimental basis for one year, a course in calculus. This course was described as follows:

"This course will devote itself to the analysis of those physical problems which are best treated by the application of the calculus. It is the purpose of the course to teach the techniques and processes of

the calculus in connection with physical phenomena and relationships, which ordinarily occur in engineering practice. In order to make this experimental course best serve its purpose, it is proposed that a committee be set up consisting of a representative of the Department of Electrical Engineering, a representative of the Department of Physics, a representative of the Department of Mathematics, and a representative of the Department of Mechanics; the chairman to be appointed by the Dean of the College of Engineering. The committee is to have full power over courses, including the appointment of instructors."

After considerable discussion it was decided to call a special meeting of the faculty of the College of Engineering to dispose of this request. This meeting was called for May 15 and at that time the Department of Mathematics presented the following substitute motion:

"In order to promote cooperation between the basic science departments and the technical departments in the College of Engineering, the Department of Mathematics wishes to present the following motion as a substitute for the motion presented to the faculty of the College of Engineering on May 9, 1934, by the Department of Electrical Engineering relative to the caculus.

1. That a committee be appointed consisting of a representative of the Department of Mathematics, a representative of the Department of Physics, a representative of the Department of Chemistry, and such other members of the faculty of the College of Engineering as the faculty wishes to name.
2. That the functions of this committee be:
   a) To determine the principles and applications to be embodied in all courses required of engineering students.
   b) To determine the proper order of presentation of topics from the standpoint of correlation with courses in physics, chemistry, mechanics, and the technical courses in engineering.
   c) To make recommendations concerning the subject matter to be contained in certain advanced courses in mathematics to satisfy the needs of advanced students in Engineering, Physics and Chemistry.
   d) To make recommendations concerning improvements in teaching.

The request of the Electrical Engineering Department was granted and the motion presented by the Department of Mathematics was passed. Two committees were set up, one to carry on the experi-

ment requested by the Department of Electrical Engineering and a larger one to carry on the work suggested by the motion presented by the Department of Mathematics. The larger committee met and decided to spend its efforts during the year 1934-5 on the special calculus course and leave the broader aspects of the problem to a later time.

Soon after it was set up the committee on the special course in calculus met and decided on a policy to be carried out during the year's work. This policy, roughly speaking, was to present the calculus as a branch of mathematics needed to solve certain physical problems that cannot be accurately or conveniently solved by any simpler means of computation. In this presentation special emphasis was to be placed on the mathematical formulation of written problems and the interpretation of results. The principles of the calculus were to be throughly learned and used in all possible physical situations.

Since the request involved only students in the Department of Electrical Engineering the committee next turned its attention to the segregation of these students and providing teachers for the sections thus made. It was found that students in the Department of Electrical Engineering were scheduled to take the calculus at three different hours and that there were not enough students at each hour to form a section. Consequently the sections were filled up with students from the Departments of Civil and Mechanical Engineering. Of the three sections thus formed, one was taught by a member of the faculty of the Department of Electrical Engineering, one by a member of the faculty of the Department of Mechanics, and the third by the writer. Dr. C. R. Wylie of the Department of Mathematics assisted in the selection of topics, the collection of suitable supplementary problems and in the actual teaching from time to time.

These four teachers met every two weeks throughout the year and discussed difficulties encountered in presenting certain topics, and various methods for overcoming these difficulties; attempted to to determine what types of physical problems were most useful at various stages of the work; made recommendations relating to future policies;and tried to evaluate the student's previous training in the light of his performance in the calculus. From these discussions it soon became apparent that

1) The student's previous training in the formal part of the freshman work in mathematics seemed fairly adequate, but that the solution of written problems needed more attention.
2) The ability to formulate physical problems in mathematical symbols was woefully weak.

This latter defect led to the suggestion that some sort of cooperative effort be undertaken by the Department of Physics and the Department of Mathematics. In this undertaking the Department of Physics was to develop the principles of physics from the standpoint of the calculus whenever practicable, and the Department of Mathematics was to furnish the proper mathematical background for such physical development. The Department of Physics agreed to try the experiment and assigned two members of its staff to help work out such a course.

It was agreed to try the experiment during the winter quarter on a single section consisting of about thirty students belonging to the trailer group. Trailers were chosen because no others were starting calculus and physics during the winter quarter. The instructors teaching mathematics and physics to this group kept in the closest possible contact during the quarter and both met with the teachers of the experimental sections of the calculus. In this new venture the topics in both the physics and the mathematics had to be arranged to a certain extent. For example, in the classes in mathematics the concept of antiderivative was introduced along with that of derivative in order that it might be used in physics in the development of the laws of falling bodies. Immediately thereafter the notion of a definite integral was introduced by the mathematics instructor and was then used by the physics instructor to solve problems involving work, energy, etc. The differentiation and integration of the trigonometric functions were taught early in order to lay the proper mathematical background for the development of the laws of uniform circular motion and of simple harmonic motion. In every instance the mathematics of the situation was introduced about a week before the physical application took place.

The results obtained in this cooperative effort with the trailer group were such as to justify the recommendation of the instructors that it be extended to include all engineering students taking the calculus, and that it continue throughout the year. This recommendation was adopted and plans were made to put it into operation at the beginning of the academic year 1935-6. In the meantime the experimental work was carried on in the three original sections. An evaluation of that work has been made by Professor Folk of the Department of Mechanics and published in the *Journal of Engineering Education*, New Series, Vol. XXVII, page 146. (October, 1936).

In accordance with the new plan all the classes in calculus and the corresponding courses in physics were run on a cooperative basis during the year 1935-6. In order to carry out the provisions of this

plan the teachers of the classes in mathematics and in physics have met regularly and discussed the topics to be included in both the calculus and the physics; the arrangements of topics that were included; the various methods that might be used to present these topics; the symbolism to be used; and any other questions which seemed to be pertinent at the moment. In the light of actual attainment in class, rearrangement of topics was made from time to time. In all this work there was the finest kind of cooperation on the part of both departments.

Since the special calculus committee was appointed for only one year and its work obviously needed more time, this committee was continued as a sub-committee of the committee authorized in the motion made by the Department of Mathematics. This larger committee then extended its activities to include two other problems.

1) The problem of what topics should be included in the work in mathematics during the freshman year, and to best train the freshman student to set up written problems in symbolic mathematical language.

2) The problem of determining what topics beyond the calculus are needed by various classes of engineering students.

As an approach to the solution of the first of these problems the committee has made the following suggestions: that the students in algebra be required to solve more written problems; that more emphasis be placed on graphs and the graphical solution of problems; that more attention be paid to work in determinants; that less time be devoted to the formal work in trigonometry; that more time be given to the study of analytic geometry, especially that part which involves the formation of an equation from given geometrical data, with probably some attention given to setting up empirical formulas. The Department of Mathematics is now at work testing out the validity of the above suggestions.

In order to find the answer to the second problem the Department of Mathematics asked members of other departments to appear before the mathematics club and present the viewpoints of their departments relative to mathematics together with a list of topics most commonly used in their advanced courses. A number of departments have responded to this request and have made helpful suggestion. In addition to this activity the Departments of Mathematics and Electrical Engineering have conducted joint seminars at which, among other things, the applications of tensor analysis to electrical engineering

problems, and the best methods for the solution of certain types of differential equations, have been considered.

This experiment has been productive of two important results; first the students have been more impressed than ever before with the value of mathematics as a tool in the solution of the problems of science and engineering. Consequently they are better able to use their mathematics in the problems which arise in their succeeding courses. Second, the members of the faculties of the technical engineering departments are more keenly aware of the difficulties encountered in the teaching of mathematics and hence have a more sympathetic attitude toward the Department of Mathematics. Where they were formerly critical and complaining, they are now anxious to do what they can to help. These results have fully justified the efforts put forth in this experiment at the Ohio State University and lead its proponents to suggest that similar experiments be tried at other institutions.

The Cause and Cure of Delinquency in College Mathematics

By GLENN JAMES
University of California at Los Angeles

Some thirty years ago the writer began a classroom search for the causes of failures of college students in mathematics and for methods of preventing them. The work was undertaken and carried out more or less informally, solely for the benefit that it might be to my own teaching. While the findings are not new, they have never been in vogue, hence are being published in the hope of arousing discussion and further experimentation along this line.

The investigation was made in freshman classes for reasons that will be evident from the context. As nearly as possible, every factor that could either contribute to failure or aid in discovering remedies was considered. Questions that were pertinent to all cases were put in a questionnaire, which grew until it filled four pages of foolscap. They explored the student's entire life from childhood up; his work, his play, his ideals, his ambitions, his associates, his social and family life, his health, his likes and dislikes especially for different studies and teachers, the supposed natural tendencies of his family, and his opinion of the quality of his former work in mathematics. The list was concluded by a few simple questions designed to test whether he understood the fundamental operations of arithmetic and had learned to think abstractly at least in simple cases. Very personal matters having to do with his morals, religion and love affairs, were left for him to volunteer upon suggestion. When a student got "down in the dumps" he was invited to my home and told that his difficulties were removable, that it was merely a matter of careful diagnosis and prescription of an appropriate remedy, just as in physical ailments. Understanding my motive the boys always cooperated, and made it easy to find what I thought were their weaknesses, and remarkably easy to remedy them.

For three years all cases of failure in my classes, which were not due to withdrawals, were salvaged and not one failed under his next teacher. Upon the basis of these results, I asked permission to take charge of all the delinquencies in our department. The head of the department replied that he could not let me do it for three reasons: first, because it would offend the other teachers; second, because the

success of the method was due to the sort of personal relation that existed between me and the students and hence could not be put into general practice; and lastly, because the class of students dealt with were not worth it. The first reason was doubtful; the second was fallacious, for all that was required was interest in the students; and the third was not only a bit cruel, but did not recognize the fact that uplifting the poorest students of a class, uplifts the entire class. This reply, coming as it did from a man usually very sympathetic with students, shows what a knotty problem the failing student has to contend with in the attitude of his instructors. All the assistance they give him, and they usually give a great deal, is of little permanent benefit unless they place high value upon him and believe in his potential ability to succeed. Mere paternalistic cramming hinders more than it aids the attainment of real scholarship.

During the next year a freshman brother roomed with me and overheard many of these treatments. One evening after a student had gone away with his case apparently solved, my brother remarked that I was not doing anything for those fellows. "They just think you are," he contended. His remark, considered in the light of the fact that my freshmen had been studying in my classroom of evenings during this investigation, raised the question as to whether or not the things I had been doing were merely getting them to work with a vigor and persistence which would have overcome their handicaps without my prescribed remedies. This contention not only seemed plausible, but it afforded an explanation for the too remarkable results that my simple remedies seemed to have been getting. Certainly if it were true, one could use remedial methods that were less personal and hence adapted to more universal application. So it seemed worth while to shift the line of attack to test out this contention.

This second phase of the investigation aimed at no changes in the students except the removal of their inferiority complexes and the instigation of the desire to master their subjects.

It was begun by calling up the most "stupid" student in my next freshman class and telling him that he had plenty of ability, although he didn't realize it, and could catch up with his class in a little while if he would put up a good fight. But it was one thing to tell him this and quite another to give him enough faith in himself to try it out. To this end I explained that any grade he had ever received was only a measure of what he was at that time and not of what he could be if he would call upon the enormous reserve which he possessed, and that anyone should be too proud to accept low grades as a reflection upon his ancestors. Then, on the positive side, I told him about my

own experiences in overcoming my supposed natural handicaps, a matter which I had been testing out extensively, and about the stupendous efforts the world's greatest men had put on their work, recounting in detail those cases which seemed to be of most interest to him. All these and many other phases of the question we discussed at length until he got up enough enthusiam to test the theory himself. And testing it took him out of the delinquent group in a very short time.

This same sort of procedure was continued for eleven years. In all this time not a student with whom this test was being made received any special instruction in the subject matter of his course. Nevertheless, not one of them did less than C grade work, while some did A work. But more important than their grades was their taking with them into practical life that faith in oneself which is necessary for success. It is worthwhile, I believe, to review some typical cases.

One student who had failed completely during the first half of his trigonometry came up to my desk with tears in his eyes after he had gotten his mid-term paper back with a zero grade. He was past thirty and could not even add one-half and one-third when he entered this class. I told him not to worry about the past for he could pass if he did well enough from then on, and that I knew he could do the work. This was the same thing I had told him time and again apparently without effect. However, that night at ten o'clock he called by phone and told me that he had been looking that trigonometry over and believed he could "do it." And he did "do it" with a spirit that nothing could stop. However, this was a little too near the end of the semester for him to catch up in all his courses. Another department "flunked him out" and it took him six years to graduate. After graduation he kept right on going and in five years was a railroad construction engineer with a salary of $3000. Mine was $1500.

Another student, who likewise could not add one-half and one-third, got a similar start early in the semester and passed from a delinquent committee subject to the best student in his college, in one semester. In his algebra he became not only the best student in his section, but made an A on his final examination, given to him by a man who vehemently opposed my philosophy of teaching.

A few years ago a former student dropped into my office and told me about his experience in last moment preparation for his doctorate examination in Geology at Stanford University. He had come up to within a week of his examination date without knowing that a reading knowledge of French and German was required. He could read French but knew no German, so things looked pretty bad for him. "Then,"

he said, "I remembered that you had gotten German in one week and had told me that I could do anything anyone else could if I would work hard enough. So, I got a lot of interlinear translations and went to work. During that week I did nothing but read German and English parallel translations, and at the end of the week, I passed my examination successfully." I told him that this was a good joke on both of us, for it was French that I had learned to read in three weeks and that I probably could not have done anything with German in three weeks, let alone one. This man had been a very weak student in my calculus class at one time and had not been one of the subjects of this investigation. He had heard the theory, then just lifted himself up by his boot-straps.

These and nine other similar cases convinced me that any normal student can get mathematics if he will work hard enough, and that he will usually work hard enough after he gets rid of his inferiority complex. However, the question remains as to whether it is worth the effort to him. Is he expending a lot of energy in a line in which he cannot hope to compete, on a time basis, with supposed natural-born mathematicians?

The third phase of this investigation undertook to answer this question by correlating with the grades earned the amount of time devoted to preparation for recititation work. This is a complicated question if one takes into account the differences in powers of concentration and the habit that some students have of thinking on their feet. However, these are factors that would tend to reduce the time of the better students. Hence, as will be seen, neglecting them introduces a sort of safety factor into our conclusion. A check on about one hundred students, constituting three of my freshman algebra classes, showed that the grades were proportional to the study time except for slight variations. These variations might have been due to errors in making up the grades, to differences in previous preparation, or to slight variations in native ability. In general, the good to excellent students reported from one and a half to three hours' preparation for each recitation hour, while the poor students reported fifteen minutes to an hour and a half. These figures, although based on too few immediate cases, are confirmed by a great many conferences with students during the last thirty years. Any teacher who will go to a little bother can explode the hoax that "good" students get their work with little effort while "poor" students work "their heads off."

The results of this long investigation have convinced me that self-confidence and work are the necessary and sufficient requirements for satisfactory scholarship in college mathematics.

Promotion and demotion in the preparatory school, along with college entrance requirements remove wide variations in the inherited capacities of the students, except possibly in very rare cases. The assumption that "poor" college students possess inborn demons that hold them down despite all they can do is a false and most depressing philosophy. In most cases the belief that "mathematics is just naturally hard for me" is what makes it hard. This conviction is generally a family prejudice but, in some cases has been started by some mishap in the early years of the student's school life, and ripened under the tutelage of the grading system. So many A's, so many B's, so many C's, so many D's, although intended to operate as a ladder for the students to climb, frequently operate as a caste system. Once down he is likely to be kept down by his own inferiority complex and by his teacher's substituting excessive drill for understanding of fundamental processes of which she thinks he is incapable. Our entire school system is so threaded through with the conviction that there are vast differences in the native abilities of students that the great versatility and potential power which the humblest mind possesses, are often left dormant. Certainly, until one can put his finger on definite inherent incapabilities he should assume that they do not exist and that any student can do anything any other can if he will work hard enough.

Of course, one meets students who do not get mathematics even when they do work hard, but the trouble is poor health, liquor, sex abuse, loss of sleep, bad eating habits, or some such physical handicap, and in rare instances extremely poor preparatory training. These are cases in which personal investigation is very necessary and often enables one to guide the student to more sensible habits of living and better methods of thinking so that work can do its magic.

Mathematical World News

Edited by
L. J. ADAMS

Professor E. J. Moulton, Northwestern University, reports the following news items from that institution: For the first semester, E. J. Moulton, Chairman of the department, was on leave of absence, and Lois W. Griffiths, assistant professor of mathematics, is on leave for the whole academic year. Dr. H. S. Wall has been promoted from an asssistant professorship to an associate professorship, and Dr. H. L. Garabedian from an instructorship to an assistant professorship. Dr. W. D. MacMillan, professor emeritus of astronomy at the University of Chicago, was lecturer in mathematics at Northwestern during the first semester, giving a course on the Dynamics of Rigid Bodies. Messrs. H. H. Campaigne and W. N. Smith are serving as assistants in mathematics. In the summer session for 1937 courses of interest to teachers and leading to a master's degree will be given by Professor E. R. Hedrick, visiting professor from the University of California at Los Angeles and by Professors E. J. Moulton, H. S. Wall and H. L. Garabedian. The summer session begins June 21 and lasts for eight weeks.

Professors Julian L. Coolidge and George D. Birkhoff of Harvard University have been made officers of the Legion of Honor.

Professor J. H. Van Vleck of Harvard University has been given the honorary degree of Sc.D. by Wesleyan University.

Columbia University, New York City, will offer these courses in summer session, 1937:

General Introduction to Modern Mathematics. Professor E. Kasner.

Geometric Transformations and Groups. Professor Kasner.

Theory of Functions of a Real Variable. Professor J. F. Ritt.

Theory of Groups of Finite Order. Professor A. B. Brown.

Differential Equations. Professor A. C. Berry.

A Mathematics Club, meeting every two weeks, is conducted jointly by the departments of mathematics of Bryn Mawr College, Haverford College, Swarthmore College and the University of Pennsylvania. Up to the end of December 1936 papers were given before this club by Professor W. L. Ayres, of the University of Michigan,

Professor F. D. Murnaghan, of Johns Hopkins University, Professor Hans Rademacher, of the University of Pennsylvania, and Professor Tullio Levi-Civita, of the University of Rome.

For three successive years the University of Pennsylvania, Bryn Mawr College and Swarthmore College have been practicing an exchange of professors. Thus Professor Dresden and Brinkmann from Swarthmore have been giving courses at the University of Pennsylvania (Calculus of Variations, Theory of Numbers). Professor Pell-Wheeler and Hedlund have also been giving courses at the University of Pennsylvania (Linear Operators, Differential Geometry). Professors Mitchell, Kline and Rademacher from the University of Pennsylvania have been giving courses at Bryn Mawr and Swarthmore.

The following new appointments and promotions are reported from the University of Pennsylvania: Professor P. A. Caris has been promoted from assistant professor to associate professor of mathematics; Professor J. A. Shohat has been promoted from assistant professor to associate professor of mathematics; Professor Hans Rademacher has been appointed assistant professor in mathematics; Dr. J. A. Clarkson has been appointed instructor in mathematics.

Professor F. H. Safford, University of Pennsylvania, has retired from active service.

The doctorate in mathematics was conferred by the University of Pennsylvania upon Miss Vivian Spencer in June, 1936. The title of her dissertation was: *Persymmetric determinant and Jacobi matrix expressions for orthogonal Tchebycheff polynomials.*

In addition to the usual elementary courses up to and including the Calculus, the following advanced courses are offered for the summer 1937 at the University of Pennsylvania:

Advanced Calculus, Theory of Finite Groups. Professor G. H. Hallett.

Differential Equations. Professor M. J. Babb.

Modern Analytic Geometry. Professor P. A. Caris.

Theory of Abstract Spaces. Dr. J. A. Clarkson.

The Oklahoma Section of the M. A. A. met in Tulsa February 5, 1937. Professor L. W. Johnson, Central State Teachers College,

Edmond, is chairman, and Professor C. E. Springer, University of Oklahoma, is secretary. The program was as follows:

1. *Logarithmic Solution of Cubic and Quartic Equations.* Professor W. T. Short, Oklahoma Baptist University.
2. *New Methods for Computing Meteor Heights.* Professor Balfour Whitney, Capitol Hill Junior College.
3. *Associativity Conditions for Division Algebras Defined by Non-Abelian Groups of Three Generators.* Professor Dora McFarland, University of Oklahoma.
4. *Traces of Three Lines in Parallel Planes.* Professor J. H. Butchart, Phillips University.
5. *On Universal Null Forms.* Professor J. C. Brixey, University of Oklahoma.
6. *Should Colleges Offer Elementary Plane Geometry for Credit?* Professor R. C. Dragoo, Southeastern Teachers College.

The fourteenth meeting of the Indiana Section of the Mathematical Association of America will be held Friday, April 30 and Saturday, May 1, 1937, at DePauw University, Greencastle, Indiana. The State Philosophical Society will hold its meeting at DePauw the same week, closing its program on Friday. The present plans call for a joint dinner meeting on Friday evening followed by an open meeting at which both a mathematician and philosopher will lecture. The program will be planned so as to be of interest to both the mathematicians and the philosophers present.

Buchhandlung Gustav Fock, 145 West 44th Street, New York City is a good source of German works on mathematics and mathematical journals. For example, they handle such series as the seventeen-volume collection of *Mathematik und ihre Anwendungen* founded by Professors Hieb and Artin. This particular group of volumes consists of monographs and textbooks on a wide range of subjects, including algebraic numbers, real functions, Fourier series, statistical mechanics, etc. In addition, the firm accepts subscriptions for the well-known periodical *Monatshefte fur Mathematik und Physik.*

Problem Department

Edited by
ROBERT C. YATES

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

SOLUTIONS

No. 130. Proposed by. *H. T. R. Aude,* Colgate University.

A man states that he was x years old in the year x^2. He adds, if to the number of my years you add the number of my month, it equals the square of the date. When was he born?

Solution by *Lucille G. Meyer,* New Orleans, Louisiana.

A man living now could not have been 43 years old in 1849, that is $(43)^2$. Therefore, the man must have been 44 years old in 1936. From the conditions given,

$$44+m=d^2$$

$$0<m<13$$

whence $m=5$ is the only integral solution, and $d=7$. The man was born May 7, 1892.

Also solved by *A. C. Briggs, Walter B. Clarke.*

No. 131. Proposed by *T. A. Bickerstaff,* University of Mississippi.

In the right triangle ABC, $(C=90°)$, equal line segments are drawn from A and B to A' and B' on the opposite sides, intersecting at O. Let the bisector of AOB' and $A'OB$ intersect $A'B$ and AB' at M and N respectively. Show that

$$CM \cdot A'B = CN \cdot AB'$$

Solution by *W. J. Howe,* Colgate University.

The relation $A'OM = B'ON$ can be expressed as follows:

$$CA'A - CMN = CB'B - CNM.$$

Taking the sine of the difference of these angles, remembering that $AA' = BB'$, there results:

$$CM(AC + CB') = CN(A'C + CB)$$

or

$$CN/CM = (2AC - AB')/(2BC - A'B) \tag{1}$$

Applying the Pythagorean theorem to the triangles $AA'C$ and $BB'C$ there results, since $AA' = BB'$:

$$A'B/AB' = (2AC - AB')/(2BC - A'B) \tag{2}$$

On comparing the relations (1) and (2), the desired result follows.

Also solved by *Karleton W. Crain, Walter B. Clarke,* and the Proposer.

Late Solutions: Nos. 112, 113, 114 by *A. C. Briggs.*

PROPOSED PROBLEMS

Note: No solutions have been received for the following problems: Nos. 30, 48, 58, 62, 81, 107.

No. 132. (Corrected) Proposed by *William E. Byrne,* V. M. I.

Let (x_0, y_0) be a point P_0 of the curve C, $y = f(x)$. Under the assumptions that $f(x)$ admits a Taylor development about the point P_0 and that $f''(x_0), f'''(x_0) \neq 0$, find

$$\lim_{x_1 \to x_0} \frac{(x_1 - x_0)^n + (x_2 - x_0)^n}{(x_1 + x_2 - 2x_0)}, \qquad (n = 2, 3, \cdots)$$

where $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are points of C and the chord P_1P_2 is parallel to the tangent P_0T_0 at P_0. What happens if $f''(x_0) = 0$, $f'''(x_0) = 0$?

No. 133. Proposed by *Nathan Altshiller-Court,* University of Oklahoma.

Find the locus of the point of contact of the spheres of a coaxial net* with a given sphere (not belonging to the net).

*Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 191. The Macmillan Company, 1935.

No. 134. Proposed by *Nathan Altshiller-Court*, University of Oklahoma.

The base ABC of the variable tetrahedron $DABC$ is fixed, the opposite vertex D varying so that

$$p(DA)^2=q(DB)^2+r(DC)^2,$$

where p, q, r are given constants. Show that the locus of D is, in general, a sphere. Show how to construct its center. Discuss the case $p=q=r=1$.

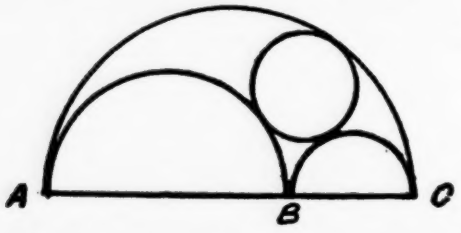
No. 135. Proposed by *Benedict C. Schwanda*, Colgate University.

Show that there exists two and only two palindromic pentagonal numbers between 1000 and 10000. A pentagonal number is of the form $n(3n-1)/2$, with n a positive integer. A palindromic number reads alike frontwards and backwards.

No. 136. Proposed by *J. S. Robberson*, Dallas, Texas.

Find the locus of the point P which moves so that the circular arc which joins P to a fixed point 0 is (1) always tangent to a fixed line through O and (2) always of fixed length a.

No. 137. Proposed by *Frank Morley*, Johns Hopkins University.



Semicircles are erected upon the given segments AB, BC, CA of a line. Find the diameter of the circle which touches the three arcs.*

*See Thomas Heath's *Works of Archimedes*; p. 307, also a letter by Professor Morley to *Nature* Vol. 139, p. 72, (January 9, 1937) and an interesting article by Prof. Frederick Soddy on *The Bowl of Integers and the Hexlet* in the same issue.—R. C. Y.

Book Reviews

Edited by
P. K. SMITH

Descriptive Geometry. By Frank W. Bubb. Macmillan Company, New York, 1935. xv+234 pages.

The text is comprehensive and well written. It is thoroughly conventional, except the method of presentation, which is logical and emphasizes the *fundamental space operations* and the *purpose of descriptive geometry* to great advantage. The fundamentals are stated and discussed clearly and concisely. The figures are plentiful, clear and accurate; and there are more well selected problems than will be used during a course. This is obviously the work of a well informed author who has had much experience in teaching this and kindred subjects.

If typographical errors exist, they did not come to my attention. Not being willing, however, to admit that any text is perfect, I suggest that the blank part of page xiv be filled with the complete Greek alphabet for the information of beginners. I also point to two slight errors in English: line 15 of Art. 3, page 11 reads "as the point on the object from whence the ray came." It should read "as the point on the object whence the ray came." Line 4 of Art. 23, page 93 reads "namely, none of the object's principal directions are parallel" should read "namely, none of the object's principal directions is parallel."

The book closes with a very good index.

Virginia Military Instiute. B. D. MAYO.

Cours de Mathematiques Speciales. Tome I. By Commissaire and Cagnac. Masson, Paris, 1936. 551 pages.

This is the first of a series of three volumes covering the mathematical preparation of students preparing for the entrance examinations of schools such as the Ecole Polytechnique and the Ecole Normale Supérieure. The first volume is divided into two books. Book I (Algebra) comprises ten chapters devoted to polynomials, permutations, combinations, the binomial theorem, the theory of irrational numbers, the theory of exponents (rational), limits, complex numbers, continuity, determinants, linear equations and forms, and the theory

of equations. Book II (Analytic Geometry) includes, besides the usual topics of plane and solid analytic geometry, a discussion of scalar and vector products, homogeneous coordinates (real and complex), systems of circles, systems of spheres, inversion, cross-ratio, projective transformations, duality, generation of surfaces. It is comparable to Graustein's *Introduction to Higher Geometry* in so far as lines, planes, circles and spheres are concerned.

The treatment throughout is characterized by careful statements and proofs of theorems and the consistent use of oriented lines and vectors. The numerous problems involve much more than mere substitution in formulas; for the most part they would appeal only to the better American students. Numerous typographical errors appear; some of them are enumerated in the errata of Volume I as found in Volumes II and III. Teachers of undergraduate courses in analytics and modern geometry could get some very useful ideas by reading Book II, especially. On the whole the book is very clear and up to date.

Virginia Military Institute. WILLIAM E. BYRNE.

A First Course in the Differential and Integral Calculus. By Walter Burton Ford. Henry Holt and Co., New York, 1937. vii+369 pp.

The object of a first course in the calculus is two-fold: first, to familiarize the student with the application of the calculus to problems of geometry and the various sciences; and secondly, to make clear the fundamental thought and concepts underlying the subject. To achieve the first objective most American textbooks employ a method of teaching which is based on the use of large numbers of illustrative examples followed by long lists of problems to be worked by the students. This method of instruction if carried to the extreme (that is, if the illustrative examples include all types of problems whose solution is left to the student) makes it possible for a student to learn to work all the exercises in a given text without understanding the fundamental principles of the subject.

"The present text," says Professor Ford in the preface, "follows in large measure the lines of development already customary in American texts on the calculus. There are 2048 examples and exercises. Of these, 179 are worked in full and serve as illustrative material complete in all particulars. Of the remainder, 92 are accompanied by both the answer and a hint as to the method of solution, 1154 are accompanied by the answer only, while, finally, 624 have neither

answer nor hint given." A commendable feature of the book is that the problems are well chosen in so far as many of them require a thorough understanding of the theory as developed in the text independently of the illustrative examples. This fact, together with the text's emphasis of the application of the calculus to physics, should enable the book to achieve to a fair degree of success the first objective stated above.

The second objective, which is the more difficult one to achieve in a first course of the calculus, has been met to some extent in the text by the presentation of the central facts and concepts in the form of distinct theorems, each followed by a proof. "As to the proofs," the author remarks in the preface, "it is well known that it is impossible in an ordinary first course in the calculus to furnish strictly rigorous proofs at all points.... The best one can do.... is to give what may be termed a 'first approximation' to a real proof. This, at least, is all that has been attempted in the present text, the student being told frankly, however, that such is the case." The clearness of the explanations of the theory, and the large number of well executed illustrative figures help the book also to meet the second objective.

In order to point out the difference between the present edition and the earlier one which appeared in 1928, we quote again from the preface. "The principle changes introduced in the present revised edition are as follows: (1) Greater flexibility has been secured by breaking up some of the longer chapters into separate chapters. Thus, there are now twenty-four chapters instead of the original seventeen, although the book is slightly less in volume than before. (2) A considerable number of new exercises have been added and eleven of the illustrative figures are either new or revised. (3) As far as possible, each page now begins either with a section heading or an exercise list, thus enhancing the general artistic effect of the whole."

The first seventeen chapters deal with the simpler topics of the differential calculus, including also partial derivatives, singular points, infinitesimals, and the polar subnormal and subtangent. Chapters XVIII to XX treat the indefinite integral which the author defines as equivalent to the primitive. In Chapter XXI the definite integral is defined as the difference between the primitive values evaluated at two different points. The area under a continuous curve is obtained by means of the limit of a sum in Chapter XXII. Thus a new expression is obtained for this area, which earlier was shown to be equal to the definite integral as defined in Chapter XXI. This is stated in the form of a theorem. The text does not give the more general definition of an integral as the limit of a sum. The student is likely to fail to

get from the book a clear conception of what the definite integral really is as used in most applied problems. Part of this chapter might have been devoted to a treatment of Duhamel's theorem and to some approximation formulas for the definite integral, such as perhaps the Prismoid Formula. This, however, would have been to the advantage of the better students only. The objection here raised applies equally well to other texts on the market. In the remainder of Chapter XXII applications of the integral are treated. The next two chapters deal with successive integration, and differential equations.

The Appendix includes a brief treatment of the method of least squares, integral tables, formulas of trigonometry, radian measurements of an angle, trigonometric tables of angles expressed in either degrees or radians at intervals of 10′ from 0° to 90°, tables of squares, cubes, square roots, and cube roots, diagrams of important curves, and an index which is not very complete. The book does not contain a table of natural logarithms, which might be considered of greater importance in a calculus book than some of the tables which are included in this text.

The book is well adapted for use in American colleges. Its material, except for the chapter on differential equations or an equivalent omission, can be covered in an eight semester hour course.

On page 144, the second last displayed formula should be followed by $x \neq 1$. On page 41, where the limit of a logarithm is taken, it might have been pointed out, for the sake of consistency with the author's statement in the preface, that it is here assumed that the limit of the logarithm is equal to the logarithm of the limit, as can be proved by more advanced methods.

College of St. Thomas. H. P. THIELMAN.

An Invitation to Mathematics. By Arnold Dresden, Professor of Mathematics, Swarthmore College. $2.80. Henry Holt and Company, New York. 1936. 453+xiii pages.

Professor Dresden's textbook with the above alluring title is planned for use in a first year of college mathematics. There is present in the book a conscious intent to depart from the usual stereotyped course in algebra, trigonometry, analytic geometry, or any of the many possible combinations of these. It was the general purpose of the author to set forth a course in mathematics which might with a minimum of technique offer an opportunity to the undergraduate for the acquisition of an understanding of what mathematics is, and the

part it plays in other fields of exact knowldge. For surely, this should be found in the training of every well educated man.

The style of the author is most attractive. The chapter, and even section headings are given in phrases which help to create an atmosphere of exciting adventure. The quotations at the opening of each chapter are testimonials to the authors' scholarly erudition. The presentation as a whole is very interesting. The eleven pages of index of names and terms demonstrates effectively the scope of this book which affords glimpses into many of the fundamental domains of mathematics.

As to the subject matter, the first steps are based on the theorem of Pythagoras, which it is assumed has been met in a course of plane geometry. From this point of departure, a detailed discussion is given of the real number system. We then pass on to the complex numbers. Elementary number theory is developed to a considerable extent, with dips into Fermat's theorem and related topics. The 'cut' definition of irrational numbers, the transcendental numbers e and π, Euclidean constructibility are topics which will epitomize the treatment of number properties. With masterly skill, the author paints a very fine picture of this domain of mathematics.

Space does not permit a detailed discussion of the many other fields of mathematics which are presented. A few more suggestive items must suffice. The modern geometry of the triangle and circle is used as a path into the discussion of the foundations of geometry and a brief treatment of non-Euclidean geometries. Slopes of curves leads into the differential calculus with a brief discussion of maxima and minima. Integration (including line and multiple) follows at once, with an excursion into linear differential equations, both ordinary and partial. The treatment closes with a chapter each on projective geometry and the map-coloring problem.

Just how much of this ambitious program can be put over with an average freshman college class, the reviewer is not prepared to state. A course such as this, must prove profoundly stimulating to the genuinely intellectually thirsty. The ideas are, in the main, very clearly put and the excellent sets of problems scattered throughout will readily test the understanding of the fundamental principles discussed. There is every reason to commend the author for a real contribution to American mathematical text books.

University of Texas. H. J. ETTLINGER.

THE SECOND DIMENSION OF ALGEBRA

The numbers of algebra correspond to the points of geometry. As an aggregate of geometrical points forms a geometrical space so an aggregation of numbers forms an algebraic space. Algebraic space, like geometrical space, may be measured in dimensions.

Algebraic space is absolute, geometrical space is relative, Zero is the absolute reference point of algebraic space. Any point, taken as origin, is the reference point of geometrical space.

Algebraic space is naturally polarized into positive and negative numbers. Geometric space is artificially polarized into positive and negative directions.

Algebraic space is naturally oriented in first, second and third dimensions. Geometric space is artificially oriented in x, y and z dimensions.

The numbers 1, 2, 3, and so on and the intervening fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and their negatives -1, -2, -3, $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{4}$, correspond to the points of a straight line and constitute the first dimension of algebra.

The square root of minus one, with symbol i, is not a number of the first dimension of algebra. The multiples of i, that is, $i1$, $i2$, $i3$ and their intervening fractions, $i\frac{1}{2}$, $i\frac{1}{3}$, $i\frac{1}{4}$ and their opposites, $-i1$, $-i2$, $-i3$, $-i\frac{1}{2}$, $-i\frac{1}{3}$, $-i\frac{1}{4}$, constitute another dimension, the imaginary dimension of algebra.

Is this imaginary dimension the second dimension of algebra? No, logically, the imaginary numbers constitute the third, not the second, dimension of algebra.

When the numbers of algebra are developed in logical order two real dimensions precede the imaginary dimension.

The FIRST dimension of algebra is formed of the multiples and fractions of unity. This is the UNITARY dimension.

The SECOND dimension of algebra is formed of the multiples and fractions of the square root of positive unity. This is the HEMITROPIC dimension.

The THIRD dimension of algebra is formed of the multiples and fractions of the square root of negative unity. This is the IMAGINARY dimension.

This thesis is developed by Robert A. Philip in a monograph BIFOLIATE NUMBERS, Price one dollar, THE MONOGRAPHIC PRESS, 106 Washington St., Fairhaven, Mass.

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